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PLANE CREMONA MAPS, EXCEPTIONAL CURVES AND ROOTS

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ABSTRACT. We address three different questions concerning exceptional and root divisors (of arithmetic genus zero and of self-intersection -1 and -2, respectively) on a smooth complex projective surface S which admits a birational morphism π to $\mathbb{P}^2$. The first one is to find criteria for the properness of these divisors, that is, to characterize when the class of C is in the W-orbit of the class of the total transform of some point blown up by π if C is exceptional, or in the W-orbit of a simple root if C is root, where W is the Weyl group acting on $\operatorname{Pic} S$; we give an arithmetical criterion, which adapts an analogous criterion suggested by Hudson for homaloidal divisors, and a geometrical one. Secondly, we prove that the irreducibility of the exceptional or root divisor C is a necessary and sufficient condition in order that $\pi_*(C)$ could be transformed into a line by some plane Cremona map, and in most cases for its contractibility. Finally, we provide irreducibility criteria for proper homaloidal, exceptional and effective root divisors.

INTRODUCTION

Consider a smooth complex projective surface S which is obtained by blowing up a cluster $T = (p_1, \ldots, p_\sigma)$ of infinitely near points in $\mathbb{P}^2$ and call $\pi_T : S \to \mathbb{P}^2$ the blowing up morphism. The classes in $\operatorname{Pic} S$ of the total transforms of a planar line and the points p_i form an *exceptional configuration* $\mathcal{E}_T = \left\{\overline{\mathcal{E}}_0, \ldots, \overline{\mathcal{E}}_\sigma\right\}$ of S ([7], [11]) and determine π_T up to a planar projectivity ([7], [8], V). Equivalently, π_T is determined up to projectivity by the collection of irreducible rational curves $E_T = \left\{\widetilde{E}_{p_1}, \ldots, \widetilde{E}_{p_\sigma}\right\}$ on S which must be contracted to reach $\mathbb{P}^2$, where $\widetilde{E}_{p_i}$ is the strict transform of the exceptional divisor of blowing up the point p_i. We call E_T an *exceptional contractile collection*. Each $\widetilde{E}_{p_i}$ is a $(-\alpha)$-curve (that is, an irreducible rational curve of self-intersection $-\alpha$), $\alpha \geq 1$, where $\alpha - 1$ equals the number of points in T proximate to p_i ([1], 1.1.26). We will say that two birational morphisms or maps to $\mathbb{P}^2$ are *distinct* if they are not in the same orbit by the action of the projective group of $\mathbb{P}^2$.

A plane Cremona map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose cluster of base points ([1], 2.1.2) is contained in T is called *dominated* by the surface S. Each plane Cremona map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ (taken up to projectivity) dominated by the surface S gives a distinct

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birational morphism $\pi_L : S \to \mathbb{P}^2$, that is, gives another cluster L of infinitely near points of $\mathbb{P}^2$ which contains the cluster of base points of ϕ^{-1}, and hence a new exceptional configuration $\mathcal{E}_L$ and a new exceptional contractile collection E_L on the surface S. Conversely, any distinct birational morphism from S to $\mathbb{P}^2$ defines by composition a distinct plane Cremona map.

For a birational morphism $\pi_T : S \to \mathbb{P}^2$, the action by a group W_σ on the group $\operatorname{Pic} S$ of divisor classes of S can be defined (see Du Val [5], Nagata [13] or [1], ch. 5). This W_σ turns out to be the Weyl group of a root system embedded in $\operatorname{Pic} S$ and has been used by several authors as Demazure [3], Manin [12], Looijenga [11], Harbourne [7] and Dolgachev-Ortland [4] to study the blowing-up of $\mathbb{P}^2$ with special incidence in the cases of the blowing-up of $\sigma \leq 8$ points in the plane or of surfaces which have an effective irreducible anticanonical divisor.

The first question we address in this paper is of arithmetical nature: to characterize the exceptional or root divisors whose classes are in the W_σ-orbit of $\overline{\mathcal{E}}_i$, $1 \leq i \leq \sigma$, or of a simple root, respectively; we call such divisors *proper*. This question is solved for homaloidal divisors, and some step is proved for exceptional divisors in [1], ch. 5. We generalize these results in section 2 for exceptional and root divisors: we give an arithmetical characterization (2.4) and prove a geometrical criterion (2.5).

In section 3 we study when an exceptional or root divisor C can be completed to make up an exceptional contractile collection. We prove that the irreducibility of C is a necessary and sufficient condition for the existence of an exceptional contractile collection of curves containing C, under the hypothesis that there is some birational morphism $\pi_L : S \to \mathbb{P}^2$ (not necessarily distinct from π_T) such that $(\pi_L)_*(C)$ is not a planar line (3.9). Moreover we show that any irreducible homaloidal, exceptional or root divisor C for which $C' = (\pi_L)_*(C) \neq 0$ can be transformed into a line $\phi_*(C')$ by some plane Cremona map ϕ dominated by the surface S (3.5). From this we derive a useful new version of Noether's factorization theorem of any plane Cremona map into de Jonquières transformations (3.6).

Finally in section 4 we provide irreducibility criteria for proper homaloidal, exceptional and root divisors, which consist of checking that they have positive intersection with the proper $(-\alpha)$-curves (4.1). Under the hypothesis that the surface S has an anticanonical nef divisor, irreducibility criteria for exceptional divisors have been given by Demazure [3] in the case of the blowing-up of $\sigma \leq 8$ points, and by Lahyane [10] in the case of $\sigma = 9$. Recently, Lahyane itself has generalized his criterion to a surface with an irreducible anticanonical divisor (author's private communication).

1. Preliminaries

In this section we introduce some definitions and we briefly recall some basic notions about Weyl groups, infinitely near points and plane Cremona maps that will be used in the sequel. The reader is referred to [3], [12], [2] and [1] for more details.

Given an integer $\sigma \geq 0$, let $\mathcal{P}_\sigma = \mathbb{Z}[E_0] \oplus \mathbb{Z}[-E_1] \oplus \cdots \oplus \mathbb{Z}[-E_\sigma]$ be an integral lattice equipped with the symmetric bilinear form $\cdot$ (intersection product) defined by $E_0 \cdot E_0 = 1$; $E_i \cdot E_i = -1$, $i > 0$; and $E_i \cdot E_j = 0$, $i \neq j$. Define $\mathbf{w}_\sigma = 3E_0 - \sum_{i=1}^{\sigma} E_i$. Let $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma) \in \mathcal{P}_\sigma$ be a vector of integral entries and fix an index $k \in \{1, \ldots, \sigma\}$ such that $\mu_k \geq \mu_i$ for any $i \in \{1, \ldots, \sigma\}$. We call n and $\mu_1, \ldots, \mu_\sigma$

the *degree* and the *multiplicities* of $\mathbf{v}$. We define the *complexity* of $\mathbf{v}$ as the rational number $j_{\mathbf{v}} = \frac{n-\mu_k}{2}$. An index $i \neq k$ such that $\mu_i > j_{\mathbf{v}}$ is called a *major index for* $\mathbf{v}$. The number of major indexes for $\mathbf{v}$ will be denoted by $h_{\mathbf{v}}$. A *top triple of indexes for* $\mathbf{v}$ is a triple of indexes $1 \leq i_1 < i_2 < i_3 \leq \sigma$ such that $\mu_{i_k} > 0$ and $\mu_{i_k} \geq \mu_j$ for any $1 \leq j \leq \sigma$, $j \notin \{i_1, i_2, i_3\}$. The vector $\mathbf{v}$ is called a $(-\alpha)$*-type*, $\alpha \in \mathbb{Z}$, if it satisfies the two equations

$$\mu_1^2 + \ldots + \mu_\sigma^2 = n^2 + \alpha\,, \tag{1.1}$$

$$\mu_1 + \ldots + \mu_\sigma = 3n + \alpha - 2\,. \tag{1.2}$$

A $(-\alpha)$-type is called *homaloidal* (respectively, *exceptional* or *root*) (cf. [1], ch. 5) if α equals -1 (respectively, 1 or 2). The root types $\mathbf{r}_0 = E_0 - E_1 - E_2 - E_3$ and $\mathbf{r}_i = E_i - E_{i+1}$, $1 \leq i \leq \sigma - 1$, are called *simple roots.* To each $\mathbf{v} \in \mathcal{P}_\sigma$ we associate the orthogonal reflection $s_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + (\mathbf{x}\cdot\mathbf{v})\mathbf{v} \in \mathbb{GL}(\mathcal{P}_\sigma)$, $\mathbf{x} \in \mathcal{P}_\sigma$. The *Weyl group* $W_\sigma \subset \mathbb{GL}(\mathcal{P}_\sigma)$ of $\mathcal{P}_\sigma$ is the group generated by the fundamental reflections $s_{\mathbf{r}_i}(\mathbf{x}) = \mathbf{x} + (\mathbf{x}\cdot\mathbf{r}_i)\mathbf{r}_i$, $0 \leq i \leq n-1$, $\mathbf{x} \in \mathcal{P}_\sigma$. We observe that W_σ leaves $\mathbf{w}_\sigma$ invariant and preserves the intersection product of $\mathcal{P}_\sigma$. For a triple of indexes $\{i, j, l\}$ we consider the orthogonal reflection $\mathbf{Q}_{i,j,l} = s_{\mathbf{v}}$, where $\mathbf{v} = E_0 - E_i - E_j - E_l$, and we call it *arithmetical quadratic transformation based on the triple* $\{i, j, l\}$ ([1], ch. 5).

Let O be a (proper) point on a smooth complex projective rational surface S. A point p *infinitely near* to O is a point lying on the exceptional divisor $E_p = \pi_p^{-1}(O)$ of the composition $\pi_p : S_p \longrightarrow S$ of a finite sequence of blowing-ups. Given two infinitely near points p, q, we write $p < q$ to mean that q is infinitely near to p. The *points preceding* p are those points p_i in the maximal chain $O = p_0 < p_1 < \ldots < p_n < p$, which are needed to be blown up to reach p. A point in a maximal chain immediately after p is called a *successor* of p or a point *in the first neighborhood* of p. The point p is called a *satellite point* if it is a singular (double in fact) point of E_p, otherwise it is called a *free point.* A point q lying on the exceptional divisor of blowing up p or on any of its successive strict transforms by further blowing-ups is called *point proximate* to p, and we denote by $q \succ p$ this relation. As it is easy to check, free points are proximate to just one point, while satellite points are proximate to exactly two points. A point p has at most two satellite successors: just one if p is free, and two if p is satellite.

A *cluster* T with origin at O is a finite set of points equal or infinitely near to O such that, for each $p \in T$, T contains all the points preceding p. A *subcluster* of a cluster T is a subset of T which is also a cluster. A pair $\mathcal{T} = (T, t)$, where T is a cluster and $t : T \longrightarrow \mathbb{Z}$ is an arbitrary map, is called a *weighted cluster* ([2], 4.1). The integer $t_p = t(p)$, for $p \in T$, is called the *virtual multiplicity* at p of $\mathcal{T}$. We also say that $\mathcal{T}$ is the cluster T *weighted by* the multiplicities $\{t_p\}_{p\in T}$. The *excess* of a weighted cluster $\mathcal{T}$ at a point $p \in T$ is the number $t_p - \sum_{q\succ p} t_q$, and the *proximity inequality* at p is $t_p - \sum_{q\succ p} t_q \geq 0$. A *consistent cluster* is a weighted cluster $\mathcal{T} = (T, t)$ that satisfies the proximity inequalities at all $p \in T$ ([2], 4.2). A cluster T is *unibranched* if any point $p \in T$ has a unique successor in T. The proximity relations between the points in a cluster T are encoded in a $T \times T$ matrix $\mathbf{P}_T = (m_{p,q})$, called *proximity matrix* of T ([2], 4.5), defined by $m_{p,p} = 1$, $m_{p,q} = -1$ if $p \succ q$, and $m_{p,q} = 0$ otherwise. In the figures the proximity relations of the points in a cluster will be described by means of Enriques diagrams (see [2], 3.9 or [1], 1.1.23); the proper points will be represented by black-filled circles and the infinitely near ones by grey-filled circles.

Let $T = (p_1, \ldots, p_\sigma)$ be a cluster of ordered points in the plane. Unless otherwise stated, the points in an ordered cluster will always be numbered following an *admissible ordering*, that is, a total ordering that refines the ordering $<$ of being infinitely near. Let $\pi_T : S_T \to \mathbb{P}^2$ be the surface obtained from $\mathbb{P}^2$ by blowing up the points in T. Given a curve C in $\mathbb{P}^2$, we denote by $\widetilde{C}$ its strict transform on S_T by π_T. A *divisor* $D \in \operatorname{Div} S_T$ is *of type* $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$ (*relative to* T) if D is linearly equivalent to $n\overline{E}_0 - \sum_{i=1}^{\sigma} \mu_i \overline{E}_i$, where, for $i > 0$, $\overline{E}_i$ is the total transform (pullback) of the point p_i and $\overline{E}_0$ is the total transform of a planar line. The classes $\overline{\mathcal{E}}_i$ of $\overline{E}_i$ in $\operatorname{Pic} S_T$ are a free $\mathbb{Z}$-module basis of $\operatorname{Pic} S_T$ and form an exceptional configuration $\mathcal{E}_T = \{\overline{\mathcal{E}}_0, \ldots, \overline{\mathcal{E}}_\sigma\}$ of the surface S_T. Hence T fixes the isomorphism f_T between $\operatorname{Pic} S_T$ and $\mathcal{P}_\sigma$ that sends $\overline{\mathcal{E}}_i$ to E_i. Since a surface may admit other birational morphisms to $\mathbb{P}^2$, the subindex T of S_T will denote the isomorphism f_T chosen when we assign a type and other related concepts to a divisor, and then we will omit the words 'relative to T'. The point p_i is a *major point for* D if the index i is a major index for $\mathbf{v}$. We call n and $\mu_1, \ldots, \mu_\sigma$ the *degree* and the *multiplicities* of the divisor D.

Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a plane Cremona map dominated by the surface S_T. The map ϕ is determined up to projectivity by a planar linear system $\mathcal{H}$ of dimension two, called a *homaloidal net*, which is the inverse image $\mathcal{H} = \phi^*|H|$ of the net of planar lines $|H|$. The type $\mathbf{v}$ of the strict transform on S_T of generic curves of $\mathcal{H}$ is called *characteristic* of the map ϕ. Let $C \in \operatorname{Div} S_T$ be a divisor of type $\mathbf{v}$; then the linear system $|C|$ on S_T is a net ([1], 2.5.2) and will also be called *homaloidal net.* The net $|C|$ defines, up to projectivity, another birational morphism $\pi_L : S_T \to \mathbb{P}^2$ which is the blowing-up of a new planar cluster L; the cluster L contains the cluster of base points of the inverse map ϕ^{-1}. The basis change matrix $\mathbf{Q} \in \mathbb{GL}(\mathcal{P}_\sigma)$ from the basis determined by $\mathcal{E}_T$ to the basis determined by $\mathcal{E}_L$ is called a *characteristic matrix* of ϕ, and it belongs to the Weyl group W_σ ([1], 5.4.18). In fact, any element of W_σ is the characteristic matrix of a plane Cremona map dominated by some surface $S_{T'}$, with the cardinal of T' equal to σ ([1], 5.4.23). Given a planar effective divisor D, the map ϕ transforms D into a planar divisor $\phi_*(D) = (\pi_L)_*(\widetilde{D})$; if $\phi_*(D) = 0$, then D is called *contractile by* ϕ. The irreducible contractile curves are called *principal curves* of ϕ, and they are the curves $(\pi_L)_*(\widetilde{E}_p)$ for any $p \in \mathcal{T}$ with positive excess, where $\mathcal{T}$ is the cluster T weighted by the multiplicities of the characteristic $\mathbf{v}$. The curves $\Theta_{p_i} = (\pi_L)_*(\overline{E}_i)$, for $p_i \in T$ which is base point of ϕ, are called *total principal curves* of ϕ.

A divisor D on a surface S is a $(-\alpha)$-*divisor* (respectively, homaloidal, exceptional or root divisor) if $D^2 = -\alpha$ and $D \cdot K = \alpha - 2$ (respectively, α equals -1, 1 or 2), where K is a canonical divisor of S. Hence if D is a $(-\alpha)$-divisor of type $\mathbf{v}$, then $\mathbf{v}$ is a $(-\alpha)$-type. An effective irreducible $(-\alpha)$-divisor (respectively, homaloidal, exceptional or root divisor) is called a $(-\alpha)$-*curve* (respectively, homaloidal, exceptional or root curve). Observe that, by the adjunction and Riemann-Roch formulae (see, for instance, [8], IV), an effective $(-\alpha)$-divisor has arithmetic genus equal to zero, and a homaloidal or exceptional divisor of positive degree is always effective.

Consider an irreducible divisor C on the surface S_T of type $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$, $n > 0$, and let $\mathcal{T}$ be the cluster T weighted by the multiplicities of C, which are in fact the effective multiplicities of the planar curve $\pi_{T*}(C)$ at the points of T. By [2], 3.5.3, the effective multiplicities $e_p(C)$ of C satisfy the proximity equalities $e_p(C) = \sum_{q \succ p} e_q(C)$ at any infinitely near point p on C, and hence $\mathcal{T}$ is consistent. Thus we can find an admissible total ordering on the weighted cluster $\mathcal{T}$ such that its virtual multiplicities follow a non-increasing sequence (cf. [1], 2.6.13).

2. Criteria for properness

In the sequel, $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$ will be a vector of integral entries, with $\mu_1 \geq \ldots \geq \mu_\sigma$, $n > 0$, and we will take $j_{\mathbf{v}} = \frac{n-\mu_1}{2}$. We say that $\mathbf{v}$ *satisfies Noether's inequality* if $\sigma \geq 3$ and $\mu_1 + \mu_2 + \mu_3 > n$. To avoid unnecessary repetition, we will write $\mathbf{v} = (n; \nu_1^{e_1}, \ldots, \nu_s^{e_s})$ to mean $\mathbf{v} = (n; \nu_1, \overset{e_1}{\ldots}, \nu_1, \ldots, \nu_s, \overset{e_s}{\ldots}, \nu_s)$ when dealing with examples.

Lemma 2.1. *Root, exceptional and homaloidal types satisfy Noether's inequality.*

Proof. Noether's inequality for exceptional and homaloidal types is classical (see [1] for a proof and classical references). The same reasoning with minor changes applies for root types (cf. [4], V.3). □

Lemma 2.2. *If there exists an irreducible curve C of type $\mathbf{v}$ on some surface S_T, then $\mu_i \leq 2j_{\mathbf{v}}$ for all $i \in \{2, \ldots, \sigma\}$.*

Proof. In case $\mu_i > 2j_{\mathbf{v}}$, we have $\mu_1 + \mu_i > n$, which contradicts the irreducibility of C. □

Lemma 2.3. *If $\mathbf{v}$ satisfies Noether's inequality, then $\mu_1 > j_{\mathbf{v}}$.*

Proof. As a direct consequence of Noether's inequality, $3\mu_1 > n$ is inferred, and hence the claim. □

We say that a root, exceptional or homaloidal type $\mathbf{v}$ *fulfills Hudson's test* if $\mu_\sigma \geq 0$ and all vectors $\mathbf{v}' = (n'; \mu'_1, \ldots, \mu'_\sigma)$ with $n' \geq 1$, given rise to by performing on $\mathbf{v}$ any finite sequence of arithmetic quadratic transformations based on top triples of indexes, have non-negative components. This adapts a test proposed in [9] by Hudson for proper homaloidal types; see [1], 5.2.15, for a proof of it. Notice that, by Noether's inequality (2.1), performing Hudson's test on a root (exceptional or homaloidal) type is a finite process giving rise to a unique finite sequence of types if we do not take into account the ordering of their last σ entries. If the root (exceptional or homaloidal) type fulfills Hudson's test, the sequence ends at $(1; 1, 1, 1, 0^{\sigma-3})$ ($(1; 1, 1, 0^{\sigma-2})$ or $(1; 0^\sigma)$, respectively).

A *homaloidal type* $\mathbf{v}$ satisfying the following equivalent conditions ([1], 5.3.4) is called *proper*:

(a) there exists a plane Cremona map whose characteristic is $\mathbf{v}$;
(b) $\mathbf{v}$ is in the W_σ-orbit of $(1; 0^\sigma)$;
(c) $\mathbf{v}$ fulfills Hudson's test;
(d) there exists a cluster T of points in $\mathbb{P}^2$ and an irreducible curve on S_T of type $\mathbf{v}$.

An *exceptional type* $\mathbf{v}$ satisfying the following equivalent conditions ([1], 5.5.15) is called *proper*:

(a) $\mathbf{v}$ is in the W_σ-orbit of $(0; 0^{\sigma-1}, -1)$, but for a permutation of the last σ entries;
(b) $\mathbf{v}$ fulfills Hudson's test.

In [1], 5.5.13, it is shown that a necessary condition for the properness of an exceptional type is that there exists a cluster T of points in $\mathbb{P}^2$ and an irreducible curve on S_T of type $\mathbf{v}$. In 2.5 we will prove that this condition is also sufficient.

We introduce the notion of properness for a root type and we will prove equivalent conditions of the same nature as for homaloidal and exceptional types. A *root type* is *proper* if it is in the W_σ-orbit of a simple root. If $\sigma \geq 4$, all simple roots are in the same W_σ-orbit, hence in this case $\mathbf{v}$ is proper if and only if it is in the W_σ-orbit of $\mathbf{r}_0 = (1;1,1,1,0^{\sigma-3})$ but for a permutation of the last σ entries. In the same way, we say that a $(-\alpha)$-*type*, $\alpha > 2$, is *proper* if it is in the W_σ-orbit of $(1;1^{\alpha+1},0^{\sigma-\alpha-1})$ or $(0;-1,1^{\alpha-1},0^{\sigma-\alpha})$ but for a permutation of the last σ entries.

An *exceptional*, *root* or *homaloidal divisor* of type $\mathbf{v}$ is *proper* if $\mathbf{v}$ is proper. Notice that the properness of a divisor on a surface S does not depend on the chosen isomorphism $f_T : \operatorname{Pic} S \cong \mathcal{P}_\sigma$, since, by the above-quoted characterization of proper homaloidal types and owing to [1], 5.4.17, any other exceptional configuration $\mathcal{E}_L$ of the surface is a W_σ-translate of $\mathcal{E}_T$ (see also [13] and [7]). Note that proper root divisors are called *real roots* in [7] and [11]. In order to make easier the comparison with previously-known results, we will also adopt the terminology of [7]: a root divisor D which is of the type of a simple root with respect to some isomorphism $f_T : \operatorname{Pic} S \cong \mathcal{P}_\sigma$, is called *sometime simple*; an effective sometime simple root divisor is called *nodal root* (cf. [11]).

Proposition 2.4. *Let* $\mathbf{v} = (n;\mu_1,\ldots,\mu_\sigma)$ *be a root type with* $n \geq 1$. *Then* $\mathbf{v}$ *is proper if and only if it fulfills Hudson's test.*

Proof. The case $\sigma = 3$ being clear, let us consider $\sigma \geq 4$. If $\mathbf{v}$ is proper, any W_σ-translate of $\mathbf{v}$ is proper as well. We claim that any proper root type $\mathbf{v}' = (n';\mu'_1,\ldots,\mu'_\sigma)$ with $n' \geq 1$ satisfies $\mu'_i \geq 0$, $1 \leq i \leq \sigma$. Indeed, by definition of properness, $\mathbf{v}' = \mathbf{r}_1\mathbf{T}$ for some $\mathbf{T} \in W_\sigma$, that is, $\mathbf{v}'$ equals the third row minus the second row of the matrix $\mathbf{T}$. Since each element of W_σ is the characteristic matrix of some plane Cremona map ([1], 5.4.23), the inequalities $\mu'_i \geq 0$, $1 \leq i \leq \sigma$, follow from [1], 3.4.4. The converse follows directly by induction on n and applying that any W_σ-transform of $\mathbf{v}$ satisfies Noether's inequality (2.1). □

As a consequence of 2.4, we derive that Hudson's test is an arithmetical criterion to decide whether a root divisor is a real root or not. Next, 2.5 gives geometrical criteria for the properness of exceptional and root types analogous to the above-quoted geometrical characterization of proper homaloidal types:

Theorem 2.5. *Let* $\mathbf{v} = (n;\mu_1,\ldots,\mu_\sigma)$ *be a root or exceptional type with* $n \geq 1$. *Then* $\mathbf{v}$ *is proper if and only if there exists a cluster* T *of points in* $\mathbb{P}^2$ *and an irreducible curve on* S_T *of type* $\mathbf{v}$.

The proof of 2.5 depends on 3.5, therefore it is deferred to section 3. In the case of a root type, 2.5 says that a root divisor C on a surface S_T is a sometime simple root if and only it there is another surface $S_{T'}$ and an irreducible divisor C' on $S_{T'}$ of the same type as C which is a nodal root.

3. Contractibility of root and exceptional curves

There are some arithmetical properties involving the multiplicities at the major indexes already known for homaloidal types (see [1], 8.2), which are also valid for exceptional and root types, as we will show next.

Lemma 3.1. *Suppose* $\mathbf{v}$ *is a root (exceptional or homaloidal) type. If there exists an irreducible curve* C *of type* $\mathbf{v}$ *on some surface* S_T*, then:*

(a) *the number of major indexes for* $\mathbf{v}$ *is* $h_{\mathbf{v}} \geq 2$*;*
(b) *if* i, j *are two different major indexes for* $\mathbf{v}$*, then* $\mu_1 + \mu_i + \mu_j > n$*;*
(c) *if* i, j, k *are three different major indexes for* $\mathbf{v}$*, then* $\mu_i < \mu_j + \mu_k$*;*
(d) *if* $m = [\frac{1}{2}h_{\mathbf{v}}]+1$*, where* $[\frac{1}{2}h_{\mathbf{v}}]$ *denotes the integral part of* $\frac{1}{2}h_{\mathbf{v}}$*, and* $\{k_1, \ldots, k_m\}$ *is a subset of major indexes for* $\mathbf{v}$*, then* $\sum_{i=1}^{m} \mu_{k_i} > \mu_1$*.*

Proof. Multiply equation (1.2) by $j_{\mathbf{v}}$, and subtract it from (1.1) to obtain

$$\sum_{i=2}^{\sigma} \mu_i(\mu_i - j_{\mathbf{v}}) = 2j_{\mathbf{v}}(\mu_1 - j_{\mathbf{v}}) + j_{\mathbf{v}}(2-\alpha) + \alpha\,.$$

Since $\alpha = 1, 2$ and $\mu_i \leq 2j_{\mathbf{v}}$ (Lemma 2.2) for any $i \in \{1, \ldots, \sigma\}$, we infer the strict inequality

$$\sum_{i=2}^{h_{\mathbf{v}}+1} (\mu_i - j_{\mathbf{v}}) > \mu_1 - j_{\mathbf{v}}\,. \tag{3.1}$$

Hence, as $\mu_1 - j_{\mathbf{v}} \geq \mu_i - j_{\mathbf{v}} > 0$ for $i \in \{2, \ldots, h_{\mathbf{v}}+1\}$, the above summation must have more than one term, that is, $h_{\mathbf{v}} \geq 2$. This proves (a). Assertion (b) is straightforward from the definitions. Assertion (c) follows from (b) and by the irreducibility of C. To see (d), consider $\{l_1, \ldots, l_{h_{\mathbf{v}}-m}\}$ the rest of major indexes for $\mathbf{v}$. Then (3.1) implies

$$\sum_{i=1}^{m}(\mu_{k_i} - j_{\mathbf{v}}) > n - 3j_{\mathbf{v}} - \sum_{i=1}^{h_{\mathbf{v}}-m} (\mu_{l_i} - j_{\mathbf{v}})\,.$$

Using $\mu_{l_i} \leq 2j$ (Lemma 2.2), which is equivalent to $-(\mu_{l_i} - j_{\mathbf{v}}) \geq -j_{\mathbf{v}}$, we infer the inequality $\sum_{i=1}^{m} \mu_{k_i} > n - 3j_{\mathbf{v}} + (2m - h_{\mathbf{v}})j_{\mathbf{v}}$. Since under our hypothesis $2m - h_{\mathbf{v}} \geq 1$, it follows that $\sum_{i=1}^{m} \mu_{k_i} > n - 2j_{\mathbf{v}} = \mu_1$, as wanted. □

Corollary 3.2. *Let* $C \in \operatorname{Div} S_T$ *be a root (exceptional or homaloidal) curve of type* $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$*,* $n > 0$*, where* $T = (p_1, \ldots, p_\sigma)$ *is numbered following an admissible ordering such that* $\mu_1 \geq \ldots \geq \mu_\sigma$*. Then*

(a) *there are at least two major points for* C*;*
(b) *any line through* p_1 *contains at most one major point;*
(c) *any major point has at most one major point proximate to it;*
(d) *there are at most* $[\frac{1}{2}h]$ *major points proximate* p_1*.*

Proof. This is a direct consequence of 3.1 and of the proximity inequalities $\mu_i \geq \sum_{p_j \succ p_i} \mu_j$, which are satisfied since C is irreducible. □

Next we focus on four types of de Jonquières plane Cremona maps depending on the proximity relations of their cluster of base points, and we study when a surface dominates one of them. They include the three possible configurations of base points for a quadratic transformation (see [1], 2.8).

Proposition 3.3. (a) *Let* $T = (p_1, p_2, p_3)$ *be a cluster of points in the plane. There exists a quadratic plane Cremona map* ϕ *whose cluster of base points is* T *(see Figure 1) if and only if the three points are not aligned and* p_2*,* p_3 *are not both proximate to* p_1*. In this case, the cluster of base points of the inverse* ϕ^{-1} *has the same proximity matrix as* T*.*

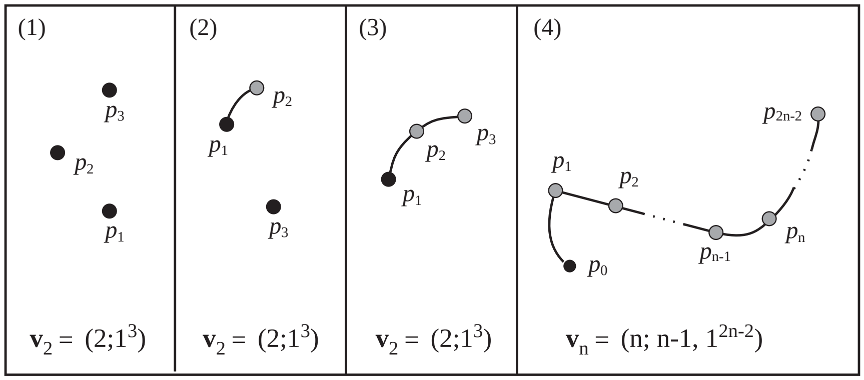

FIGURE 1. Four types of configurations of the base points of de Jonquières plane Cremona maps, which act as fundamental pieces in the factorization of plane Cremona maps.

(b) *Let $T = (p_0, \ldots, p_{2n-2})$ be a unibranched cluster of points in the plane such that the points $p_1, \ldots, p_{n-1}$ are proximate to p_0 and the points $p_n, \ldots, p_{2n-2}$ are free (see Figure 1). Then there exists a de Jonquières plane Cremona map ϕ of characteristic $\mathbf{v}_n = (n; n-1, 1^{2n-2})$ whose cluster of base points is T. Moreover, the cluster of base points of ϕ^{-1} has the same proximity matrix as T.*

Proof. We will apply the criterion for homaloidal nets of [1], 8.1.2, which consists of checking that Bézout's theorem exhibits no irreducible component of the linear system determined by the characteristic $\mathbf{v}_n$ and of checking the consistency of the multiplicities of $\mathbf{v}_n$ weighted on the cluster T. Case (a) for $\mathbf{v}_2$ is straightforward. Let us check the conditions of the criterion for case (b). The multiplicities of $\mathbf{v}_n$ are clearly consistent on T. Let us show that any irreducible plane curve C of degree $d < n$, whose strict transform $\widetilde{C} \in \operatorname{Div} S_T$ is of type $(d; m_0, \ldots, m_{2n-2})$, satisfies $nd - (n-1)m_0 - \sum_{i=1}^{2n-2} m_i \geq 0$. Since the case when C is a line is clear, assume $d > m_0$. The proximity inequalities for C are $m_0 \geq \ldots \geq m_{2n-2}$, $m_0 \geq m_1 + \ldots + m_{n-1}$. Suppose $m_{n-1} > 0$; then $m_0 \geq n-1$, which contradicts our hypothesis $n > d > m_0$. Hence $m_{n-1} = \ldots = m_{2n-2} = 0$ and thus $nd - (n-1)m_0 - \sum_{i=1}^{2n-2} m_i \geq n(d - m_0) > 0$. Thus there exists a de Jonquières plane Cremona map ϕ of characteristic $\mathbf{v}_n = (n; n-1, 1^{2n-2})$ whose cluster of base points is T.

In [1], 2.8, it is proved that the clusters of a quadratic plane Cremona map and its inverse have the same proximity matrix. For the case of a de Jonquières map ϕ of the type of Figure 1, (4) we shall apply [1], 7.3.3, which infers the proximity relations of the cluster of base points of the inverse ϕ^{-1} from some inclusion relations between the total principal curves. Since the total principal curves of ϕ^{-1} are $\Theta_{q_{2n-2}} = \ldots = \Theta_{q_1} = l$ and $\Theta_{q_0} = (n-1)l$, where l is the line joining p_0 and p_1, the cluster of base points of ϕ^{-1} satisfies the proximity relations $q_i \succ q_{i-1}$, for $i \in \{1, \ldots, 2n-2\}$, and $q_i \succ q_0$, for $i \in \{1, \ldots, n-1\}$, which are same as those of T, as we wished to show. □

Lemma 3.4. *Let C be a plane curve and suppose that $\widetilde{C} \in \operatorname{Div} S_T$ is of type $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$, where $T = (p_1, \ldots, p_\sigma)$ is numbered following an admissible* *ordering such that* $\mu_1 \geq \ldots \geq \mu_\sigma$. *Fix an integer* m, $2 \leq m \leq \frac{1}{2}\sigma + 1$, *and assume that there exists a de Jonquières plane Cremona map* ϕ *dominated by* S_T *of characteristic* $\mathbf{v}_m$ *of one of the four types of Figure 1:*

(a) *If* C *is the line joining* p_1 *and* p_2, *then* ϕ *contracts* C *to a point.*

(b) *If* C *is a line through* p_1 *and no other point of* T *lies on* C, *then* ϕ *transforms* C *into a line* $\phi_*(C)$.

(c) *If* $h_{\mathbf{v}} \geq 2$ *and* $2 \leq m \leq \frac{1}{2}h_{\mathbf{v}} + 1$, *then* ϕ *transforms* C *into a curve* $\phi_*(C)$ *of degree strictly less than* n.

(d) *The only irreducible curves which are contractile by* ϕ *are the line* p_1p_2 *and, if they exist, the lines* p_1p_3 *and* p_2p_3.

Proof. By [1], 2.9.5, the degree of the curve $\phi_*(C)$ is equal to

$$nm - \mu_1(m-1) - \sum_{i=2}^{2m-1} \mu_i = n - \sum_{i=2}^{2m-1} (\mu_i - j_{\mathbf{v}})\,.$$

Hence assertions (a) and (b) clearly follow. In the way we have chosen m for assertion (c), the points $p_2, \ldots, p_{2m-1}$ are major points for C and hence $\mu_i - j_{\mathbf{v}} \geq \frac{1}{2}$, for any $i \in \{2, \ldots, 2m-1\}$; thus

$$n - \sum_{i=2}^{2m-1} (\mu_i - j_{\mathbf{v}}) \leq n - \frac{1}{2}(2m-2) = n - (m-1) < n\,,$$

as desired. Finally, to see (d), we recall that the irreducible contractile curves by ϕ are the principal curves of the inverse map ϕ^{-1} ([1], 2.3.2). A principal curve C of ϕ^{-1} comes from a base point of ϕ^{-1} with positive excess, and this excess is precisely the degree of C ([1], 2.2.4). By direct inspection, the weighted cluster of base points of any de Jonquières map of any of the four types of Figure 1 has excess zero or one at each point, hence the claim. □

The first step towards the contractibility of root and exceptional curves is to show that they can be transformed into a line by some plane Cremona map.

Proposition 3.5. *Let* C *be an irreducible plane curve and suppose that* $\widetilde{C} \in \operatorname{Div} S_T$ *is a root* (*exceptional or homaloidal*) *curve of type* $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$, $n > 0$, *where* $T = (p_1, \ldots, p_\sigma)$ *is numbered following an admissible ordering such that* $\mu_1 \geq \ldots \geq \mu_\sigma$. *Then there exists a plane Cremona map* Φ *dominated by* S_T *that transforms* C *into a line* $\Phi_*(C)$.

Proof. Consider the subcluster $T' = (p_1, \ldots, p_{h_{\mathbf{v}}+1}) \subset T$ which comprises the point p_1 of highest multiplicity of C (which must be a proper point in the plane) and the major points for C. By 3.2(a), $h_{\mathbf{v}} \geq 2$. We will proceed by induction on the degree n of C. At each step we distinguish two cases:

(1) Some major point p_i is either proper or a free successor of some other major point. Then there exists another major point p_j which is either proper, or lies on the first neighborhood of either p_1 or p_i. Since p_1, p_i and p_j are not aligned (3.2(b)), from 3.3 it follows that there exists a quadratic plane Cremona map ϕ whose cluster of base points is (p_1, p_i, p_j). By 3.4, ϕ transforms C into a curve $\phi_*(C)$ of degree strictly less than n.

(2) All the major points are infinitely near to p_1 and no one is free in the second neighborhood of p_1. By 3.2(c), T' is a union $\bigcup_{i=1}^{s} Q_i$ of unibranched subclusters with origin p_1, where s is the number of points in T' belonging to

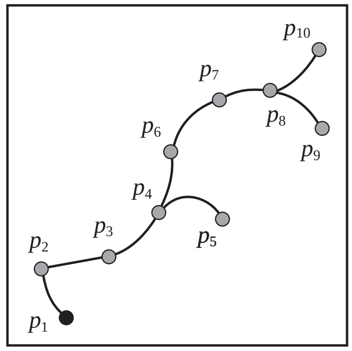

FIGURE 2. Cluster of points T in $\mathbb{P}^2$ appearing in 3.7.

the first neighborhood of p_1 and, for each Q_i, the first a_i points of Q_i (after p_1) are proximate to p_1 and the rest of its b_i points are free. Since from 3.2 (a) there are at most $[\frac{1}{2}h_{\mathbf{v}}]$ major points proximate to p_1, it follows that for at least one Q_i we have $a_i \leq b_i$. Therefore, applying 3.3, there exists a de Jonquières plane Cremona map ψ of characteristic $\mathbf{v}_{a_i+1}$ whose cluster of base points comprises the first $2a_i+1$ points of Q_i. By 3.4, ψ transforms C into a curve $\psi_*(C)$ of degree strictly less than n.

Now, by the hypothesis of induction, the claim follows. □

In the language of roots, 3.5 says that any irreducible effective root is a nodal root. A direct consequence of 3.5 is the refined version of Noether's factorization theorem (Corollary 3.6). It refines [1], 8.3.4, in the sense that it reduces the number of de Jonquières maps which act as fundamental pieces in the factorization. This easily gives the characterization 3.8 of surfaces which do not dominate any plane Cremona map, which is used to prove the contractibility of the exceptional and root curves in 3.9. Other available modern proofs of Noether's factorization theorem in the literature (as for instance that of Shafarevich [14]), which use quadratic plane Cremona maps in the factorization, have the drawback that these quadratic maps are not necessarily dominated by the surface of departure.

Corollary 3.6 (Noether's factorization theorem)**.** *Let ϕ be a plane Cremona map dominated by a surface S_T. Then ϕ factorizes into a sequence of de Jonquières maps, each one dominated by S_T and of one of the four types of Figure 1.*

Remark 3.7. Applying 3.6, we obtain by composition an inductive procedure to describe all the plane Cremona maps, modulus projectivity, dominated by a surface. We have illustrated this fact with the configuration of infinitely near points of Figure 2: the only plane Cremona maps dominated by the surface S_T are those whose characteristic is either $(3;2,1^4,0^5)$, $(3;2,1^3,0,1,0^4)$, $(6;4,2^3,0,2,1^3,0)$ or $(6;4,2^3,0,2,1^2,0,1)$.

From 3.6 directly follows a characterization of clusters T of points in the plane which admit no plane Cremona map dominated by S_T other than a projectivity, that is, surfaces S_T which have only one birational morphism to $\mathbb{P}^2$ up to projectivity.

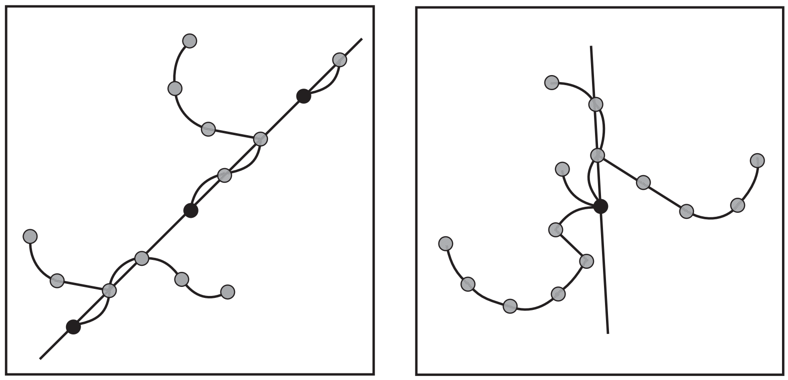

FIGURE 3. Examples of clusters T of points in $\mathbb{P}^2$ which do not admit any plane Cremona map dominated by S_T (lines through more than two points are drawn). In both examples, any cluster T' union of unibranched subclusters appearing in the figure with origin at a proper point does not admit any plane Cremona map dominated by $S_{T'}$.

Corollary 3.8. *Let T be a cluster of points in the plane. There is no plane Cremona map dominated by S_T if and only if one of the following conditions holds* (*cf. Figure 3*):

(a) *there is a single proper point p in T; any successor $q \in T$ of p has at most one free successor $q' \in T$ which lies on the line pq; any unibranched subcluster $Q \subset T$ with origin at p of cardinal $a + b + 1$ such that its first a points after p are proximate to p and the rest of its b points are free satisfies $a > b$;*

(b) *there are at least two proper points in T; there is a line l containing all the proper points $p_1, \ldots, p_s$ of T; any p_i has at most one successor $q_i \in T$ which lies on l; any q_i has at most one free successor in T which lies on l as well; any unibranched subcluster $Q_i \subset T$ with origin p_i of cardinal $a_i + b_i + 1$ such that its first a_i points after p_i are proximate to p_i and the rest of its b_i points are free satisfies $a_i > b_i$.*

Now we have the tools to prove a more sharpened version of 3.5 for root and exceptional curves, namely that if the root or exceptional curve can be transformed into a curve of degree $n > 1$ by some plane Cremona map, then it is contractile by some other plane Cremona map:

Theorem 3.9. *Let C be an irreducible plane curve of degree $n > 1$ and suppose $\widetilde{C} \in \operatorname{Div} S_T$ is an exceptional or root curve. Then there exists a plane Cremona map ϕ dominated by S_T for which C is a principal curve, that is, there exists a birational morphism π_L from S_T to $\mathbb{P}^2$, which contracts C to a point $q \in L$, where L is the cluster of points blown-up by π_L. Furthermore this plane Cremona map ϕ may be chosen in such a way that if $\widetilde{C}$ is an exceptional curve, then q is a proper point in $\mathbb{P}^2$ to which no other point of L is infinitely near, and if $\widetilde{C}$ is a root curve, then q is a proper point in $\mathbb{P}^2$ to which a single point in L is infinitely near.*

Proof. Applying 3.5, there is a plane Cremona map ψ dominated by S_T and there is a cluster L' in $\mathbb{P}^2$ with $S_{L'} = S_T$ such that $\psi_*(C)$ is a line l which goes either through exactly two points p_1, p_2 of L' if $\widetilde{C}$ is exceptional, or through exactly three points p_1, p_2, p_3 of L' if $\widetilde{C}$ is root. Assume the points p_i are numbered following an admissible ordering. From the proof of 3.5, we know that $\psi = \psi_1 \circ \ldots \circ \psi_s$ is a composition of de Jonquières maps, each ψ_i dominated by $S_{L'}$ and of one of the four types of Figure 1, and that the degree of $(\psi_1 \circ \ldots \circ \psi_{i+1})_*(C)$ is strictly less than the degree of $(\psi_1 \circ \ldots \circ \psi_i)_*(C)$. Hence $(\psi_1 \circ \ldots \circ \psi_{s-1})_*(C) = (\psi_s^{-1})_*(l)$ has degree $n' > 1$. By 3.3, ψ_s^{-1} is of the same type as ψ_s, that is, ψ_s^{-1} is a de Jonquières map dominated by $S_{L'}$ whose cluster of base points has the same proximity matrix as the cluster of base points of ψ_s. We claim that there is some proper point $p \in L'$ not lying on l. Indeed, if all points in L' are infinitely near to some p_i, then, by 3.4, any plane Cremona map dominated by $S_{L'}$ of one the four types of Figure 1, and in particular ψ_s^{-1}, transforms l into line or contracts it to a point, which contradicts the fact that $(\psi_s^{-1})_*(l)$ has degree $n' > 1$. Now the quadratic plane Cremona map τ whose base points are (p_1, p_2, p) gives a new cluster L of $\mathbb{P}^2$ with $S_L = S_T$ and contracts l to a proper point $q \in L$ of $\mathbb{P}^2$. A point which is a successor of q in L comes from a point of L' different from p_i and lying on the line l. Thus if $\widetilde{C}$ is exceptional, then no point of L is infinitely near to q, and if $\widetilde{C}$ is root, then there is a unique point of L which is infinitely near to q. □

We can now prove 2.5:

Proof of 2.5. Assume $\mathbf{v}$ is proper. This implication for $\mathbf{v}$ exceptional is [1], 5.5.14, so we may restrict to the case where $\mathbf{v}$ is a root type. The claim is clearly true for $\sigma = 3$, so suppose $\sigma \geq 4$. Let $\mathbf{T} = \mathbf{Q}_1 \ldots \mathbf{Q}_s \in W_\sigma$ be the product of arithmetical quadratic transformations that appear when performing Hudson's test for $\mathbf{v}$; in particular $\mathbf{vT} = \mathbf{r}_0$. Let $\mathbf{u}$ be the homaloidal type of the first row of $\mathbf{T}^{-1} = \mathbf{Q}_s \ldots \mathbf{Q}_1$. Consider the non-empty Zariski-open subset $U \subset \mathbb{P}^2 \times \overset{\sigma}{\ldots} \times \mathbb{P}^2$ constructed in the proof of [1], 5.2.19, for $\mathbf{u}$. By the definition of Husdson's test, the degree of the type $\mathbf{r}_0\mathbf{Q}_s \ldots \mathbf{Q}_i$ is strictly greater than the degree of $\mathbf{r}_0\mathbf{Q}_s \ldots \mathbf{Q}_{i+1}$, $s \geq i \geq 1$, and in particular strictly greater than one. Hence the alignment of the first three points p_1, p_2 and p_3 of $(p_1, \ldots, p_\sigma) \in \mathbb{P}^2 \times \overset{\sigma}{\ldots} \times \mathbb{P}^2$ does not affect the recursive construction of U. Therefore the points $(p_1, \ldots, p_\sigma) \in U$ such that p_1, p_2, p_3 are aligned and satisfy no other alignment condition form a non-empty subset V of U. Take a cluster of points $L = (p_1, \ldots, p_\sigma) \in V$ and let l be the line joining p_1, p_2 and p_3. Still by [1], 5.2.19, we can consider $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a plane Cremona map whose characteristic is $\mathbf{u}$ and is dominated by S_L. Then φ determines a new cluster T with $S_L = S_T$ and the strict transform $\widetilde{l}$ of l by π_L is an irreducible curve on S_T of type $\mathbf{v}$. The converse follows from 3.9. □

4. Irreducibility criteria

Given a proper homaloidal divisor $C \in \operatorname{Div} S_T$, the dimension of $|C|$ is not less than two by the Riemann-Roch formula. Any result which characterizes when $|C|$ is a homaloidal net is in fact an irreducibility criterion, due to the result of Enriques [6], libro V.II.20 (cf. [1], 5.1.1), that says that $|C|$ is a homaloidal net if and only if generic curves in $|C|$ are irreducible. In [1], 8.1.2, a criterion for homaloidal nets is given, which essentially is Harbourne's [7], th. (1.1). Next 4.1 sharpens this result and presents analogous irreducibility criteria for exceptional and root divisors.

Theorem 4.1. (a) *Let $C \in \operatorname{Div} S_T$ be a proper exceptional divisor. Then C is irreducible if and only if $C \cdot D \geq 0$ for any proper $(-\alpha)$-curve D of S_T, $\alpha \geq 2$.*

(b) *Let $C \in \operatorname{Div} S_T$ be a proper homaloidal divisor. Then generic curves in $|C|$ are irreducible if and only if $C \cdot D \geq 0$ for any proper $(-\alpha)$-curve D of S_T, $\alpha \geq 2$.*

(c) *Let $C \in \operatorname{Div} S_T$ be an effective proper root divisor. Then C is irreducible if and only if $C \cdot D \geq 0$ for any $(-\alpha)$-curve D of S_T, $\alpha \geq 2$, different from C.*

Proof. It is clear in all three cases that the conditions are necessary for the irreducibility. Let us show that they are also sufficient. Since the result is clearly true if C belongs to a section of some divisor of the exceptional configuration $\mathcal{E}_T$, we may suppose that C is of type $\mathbf{v} = (n; \mu_1, \ldots, \mu_\sigma)$, $n > 0$, and that the points in $T = (p_1, \ldots, p_\sigma)$ are numbered following an admissible ordering such that $\mu_1 \geq \ldots \geq \mu_\sigma$. We will exhibit a birational morphism π_L from S_T to $\mathbb{P}^2$, that is, a cluster L with $S_T = S_L$, which has the property that $C \in \operatorname{Div} S_L$ is the class of a planar line. We will proceed by induction on n. Since C is proper, all the arithmetical properties from (a) to (d) stated in 3.1 of the type $\mathbf{v}$ are still valid, by 2.5. Let us show that, under our hypothesis, the geometrical properties of 3.2 hold as well. We claim that the cluster (T, μ) weighted by the multiplicities μ_i of the type $\mathbf{v}$ is consistent. Indeed, if (T, μ) is not consistent, then $\mu_i - \sum_{p_j \succ p_i} \mu_j < 0$ for some p_i. This means that $C \cdot \widetilde{E}_{p_i} < 0$, where $\widetilde{E}_{p_i}$ is the strict transform on S_T of the exceptional divisor of blowing up p_i. Since $\widetilde{E}_{p_i}$ is a $(-\alpha)$-curve, where $\alpha - 1$ equals the number of points in T proximate to p_i, the above inequality contradicts our hypothesis. The consistence of (T, μ) together with 3.1 give assertions (c) and (d) of 3.2. It remains to see that 3.2(b) also holds. Indeed, if some line l though p_1 contains α major points, $\alpha \geq 2$, then l is a $(-\alpha)$-curve and $C \cdot \widetilde{l} < 0$, by 3.1(b), against our hypothesis. Now the same reasoning as in the proof of 3.5 applies: there exists a plane Cremona map ϕ whose cluster of base points is contained in the subcluster of major points of C. This map ϕ determines a new cluster L' of $\mathbb{P}^2$ such that $S_T = S_{L'}$ and that $C \in \operatorname{Div} S_{L'}$ is of type $\mathbf{v}' = (n'; \mu'_1, \ldots, \mu'_\sigma)$ with $n' < n$. By hypothesis of induction, there exists a cluster $L = (q_1, \ldots, q_\sigma$ with $S_L = S_T$ such that $C \in \operatorname{Div} S_L$ is of type $(1; 1, 1, 0^{\sigma-2})$, $(1; 0^\sigma)$ or $(1; 1, 1, 1, 0^{\sigma-3})$, in case C is exceptional, homaloidal or root, respectively. If C is homaloidal, then we are done, since $|C|$ is precisely the homaloidal net of the plane Cremona map composition of the different maps obtained in the inductive steps. If C is exceptional, then it remains only to check that the class $\overline{\mathcal{E}}_0 - \overline{\mathcal{E}}_1 - \overline{\mathcal{E}}_2 \in \operatorname{Pic} S_L$ has an irreducible section, that is, that the only points of L lying on the line ℓ are q_1 and q_2, $\ell = q_1 q_2$. Indeed, if ℓ contains $\beta > 0$ other points of L, then ℓ is a $(-\beta-1)$-curve and $C \cdot \widetilde{\ell} = -1$, against our hypothesis. If C is a root, then it can be checked in an analogous way that the class $\overline{\mathcal{E}}_0 - \overline{\mathcal{E}}_1 - \overline{\mathcal{E}}_2 - \overline{\mathcal{E}}_3 \in \operatorname{Pic} S_L$ has an irreducible section, and we are done. □

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